Oscillation of the lever caused by the swinging of the pendulum

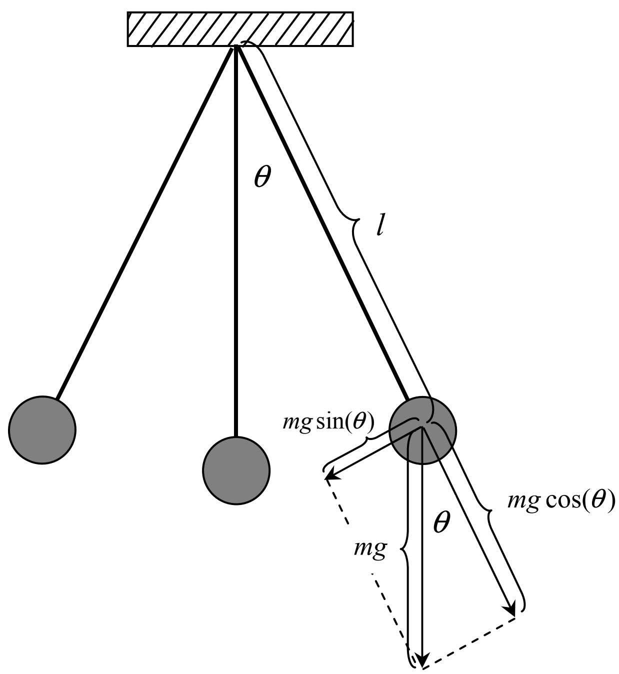
Introduction

Two-armed lever has a weight on one arm, and the pendulum on the other, which is oscillating forcibly because of the additional force moment created by the swinging of the pendulum. “Gradual“ oscillations of the lever are also possible, but for the selection of the masses on lever arms and the selection of the force arm and the weight arm, there are no spontaneous oscillations.

The first part will contain the theory of the mathematical pendulum, and then all the results will be used for the solution of the basic problem: oscillation of the lever due to the oscillation of the pendulum.

1. Swinging of the mathematical pendulum

Swinging of the pendulum under the influence of the active component of the weight:



Picture 1.

Equation for the swinging of the pendulum is:

$$m\ddot{s} = -mg\sin(\theta), \qquad s = l\theta \tag{1}$$

From that, it follows:

$$\begin{aligned} ml\ddot{\theta} &= -mg\sin(\theta) \\ \ddot{\theta} + \frac{g}{l}\sin(\theta) &= 0 \end{aligned} \tag{2}$$

It is presumed that the oscillations are small, so the approximation is:

$$\sin(\theta) \approx \theta \tag{3}$$

Therefore, differential equation (2) becomes:

$$\ddot{\theta} + \Omega^2 \sin(\theta) = 0\,,\ \Omega = \sqrt{\frac{g}{l}} \tag{4}$$

Equation (4) will be solved for border cases:

$$\theta(0) = 0,\ \theta(\frac{\pi}{2\Omega}) = \theta_0 \tag{5}$$

In the domain of small oscillation which is being researched, it could be considered that $\theta_0 = \frac{\pi}{6} = 0.523$, because $m\frac{\pi}{6} = 0.5$, so the approximation (3) is correct.
Solution for the equation (4) is:

$$\theta(t) = A\cos(\Omega t) + B\sin(\Omega t) \tag{6}$$

Therefore:

$$\theta(0) = 0 = A,\ \theta(\frac{\pi}{2\Omega}) = A\cos(\frac{\pi}{2}) + B\sin(\frac{\pi}{2}) = B = \theta_0 \tag{7}$$

And,

$$\theta(t) = \theta_0 \sin(\Omega t) = \theta_0 \sin(\sqrt{\frac{g}{l}}t) \tag{8}$$

Oscillation period is:

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{l}{g}} \tag{9}$$

So that (8) can be shown like this:

$$\theta(t) = \theta_0 \sin(\frac{2\pi}{T}t) \tag{10}$$

At the end, we will get the oscillating speed and the middle value of its square. This way, we can get medium centrifugal force which affects the pendulum during swinging. The differentiation (10) gives us:

$$\dot{\theta}(t) = \frac{2\pi}{T}\theta_0 \cos(\frac{2\pi}{T}t) \tag{11}$$

To get the tangential speed, we multiply the corner speed with L and get:

$$v(t) = l\,\dot{\theta}(t) = \frac{2\pi l}{T}\theta_0 \cos(\frac{2\pi}{T}t)$$

And we also get:

$$v^2(t) = \frac{4\pi^2 l^2}{T^2}\theta_0^{\,2} \cos^2(\frac{2\pi}{T}t) \tag{12}$$

From that, we get:

$$\bar{v}^2 = \frac{4\pi^2 l^2}{T^2}\theta_0^{\ 2}\,\frac{4}{T}\int_0^{T/4}\cos^2(\frac{2\pi}{T}t)dt \tag{13}$$

And we have:

$$\cos^2(\frac{2\pi}{T}t) = \frac{1}{2}(1+\cos(\frac{4\pi}{T}t))$$

After that, we get:

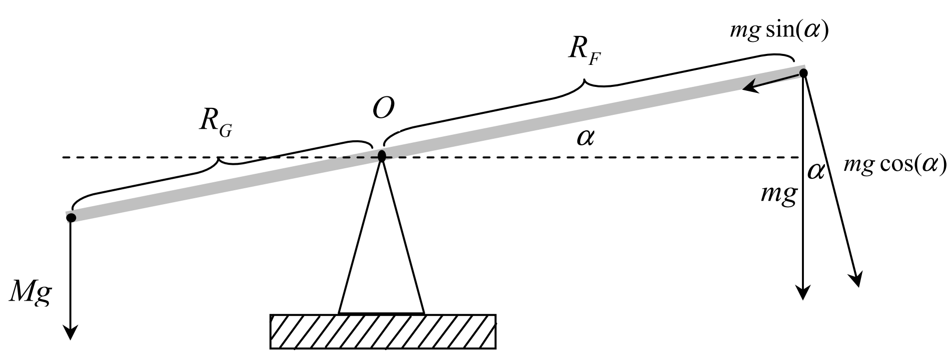
$$\bar{v}^2 = \frac{4\pi^2 l^2}{T^2}\theta_0^{\ 2}\cdot\frac{4}{T}(\frac{T}{8}+\frac{1}{\pi}\sin(\frac{4\pi}{T}t)\Big|_0^{T/4} = \frac{2\pi^2 l^2}{T^2}\theta_0^{\ 2} = 2\pi^2 l^2\theta_0^{\ 2}\,\frac{g}{4\pi^2 l^2} = \frac{1}{2}\theta_0^{\ 2} gl$$

Since the centrifugal force is $\bar{F}_z = m\frac{\bar{v}^2}{l}$, the final result is:

$$\bar{F}_z = \frac{1}{2}\theta_0^{\ 2} mg \tag{13}$$

2. Oscillation of the lever under the influence of the swinging of the pendulum

The point A is the weight with mass *M*, and the point B is the pendulum with mass *m*.



Picture 2.

Forced oscillations of the lever can be caused under the influence of the moment of the force (see picture 2):

$$M = mgR_F\cos(\alpha) \tag{14}$$

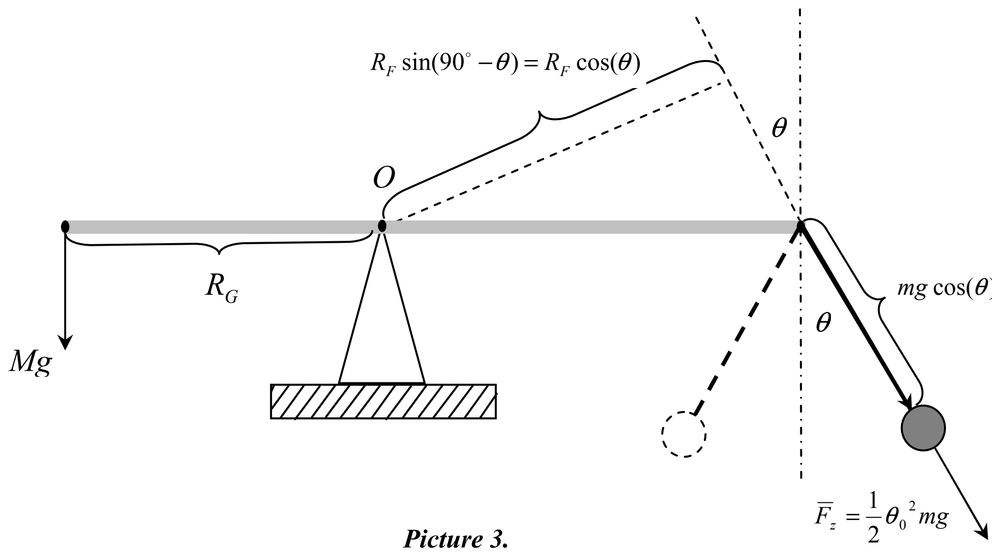
The equation of free oscillations would be:

$$J\ddot{\alpha} + mgR_F\cos(\alpha) = 0 \tag{15}$$

Where J is the moment of the inertia of the lever part OAC. In further text, we will approximate:

$$J = mR_F^{\ 2} \tag{16}$$

The pendulum creates an additional moment of the force by swinging.



Picture 3.

We get the moment of force due to swinging of the pendulum, as in the picture where the force is $mg\cos(\theta)$ and $\overline{F}_z$ is:

$$\mu_{ad} = (mg\cos(\theta) + \overline{F}_z)R_F \sin(90^\circ - \theta) = (mg\cos^2(\theta) + \frac{1}{2}\theta_0^{\,2} mg\cos(\theta))R_F$$

Total moment of the force created due to the swinging of the pendulum is:

$$\mu_{TOT} = MgR_G - \mu_{ad} = MgR_G (mg\cos^2(\theta) + \frac{1}{2}\theta_0^{\,2} mg\cos(\theta))R_F$$

This equation of the forced lever osculation (it is forced because of the pendulum oscillations) we get:

$$J\ddot{\alpha} + mgR_F\cos(\alpha) = MgR_G - mgR_F\cos^2(\theta) - \frac{1}{2}\theta_0^{\,2} mgR_F\cos(\theta)$$

That is:

$$\ddot{\alpha} + \frac{mgR_F}{J}\cos(\alpha) = \frac{MgR_G}{J} - \frac{mgR_F}{J}\cos^2(\theta) - \frac{1}{2}\theta_0^{\,2}\frac{mgR_F}{J}\cos(\theta) \qquad (17)$$

In further text we will use a series of approximations. If we take the approximation (16), then (17) becomes:

$$\ddot{\alpha} + \frac{g}{R_F}\cos(\alpha) = \frac{M}{m} g \frac{R_G}{R_F^{\,2}} - \frac{g}{R_F}\cos^2(\theta) - \frac{1}{2}\theta_0^{\,2}\frac{g}{R_F}\cos(\theta) \qquad (18)$$

Next series of approximations is based on the fact that angles α and θ are small.
We will take:

$$\cos(\alpha) \approx 1 - \frac{1}{2}\alpha^2 \approx 1; \quad \cos^2(\theta) \approx (1 - \frac{1}{2}\theta^2)^2 \approx 1 - \theta^2; \ \cos(\theta) \approx 1 - \frac{1}{2}\theta^2 \qquad (19)$$

Include (18) into (19) and we will get:

$$\ddot{\alpha} = -\frac{g}{R_F} + \frac{g}{R_F}\frac{M}{m}\frac{R_G}{R_F} - \frac{g}{R_F} - \frac{\theta_0^{\,2}}{2}\frac{g}{R_F} + \frac{g}{R_F}\theta^2 + \frac{1}{2}\theta_0^{\,2}\frac{g}{R_F}\frac{1}{2}\theta^2$$

Or:

$$\ddot{\alpha} = \frac{g}{R_F}(1+\frac{1}{4}\theta_0{}^2)\theta^2 - \frac{g}{R_F}(2+\frac{\theta_0{}^2}{2}-\frac{M}{m}\frac{R_G}{R_F}) \tag{20}$$

Then, based on (9):

$$\theta^2 = \theta_0{}^2 \sin^2(t\sqrt{\frac{g}{l}}) \tag{21}$$

Include that into (20) and we will get:

$$\ddot{\alpha} = \frac{g}{R_F}\theta_0{}^2(1+\frac{1}{4}\theta_0{}^2)\sin^2(t\sqrt{\frac{g}{l}}) - \frac{g}{R_F}(2+\frac{\theta_0{}^2}{2}-\frac{M}{m}\frac{R_G}{R_F}) \tag{22}$$

The equation (22) is integrated by time with initial conditions:

$$\alpha(0)=\alpha_0,\ \dot{\alpha}(0)=\alpha_0 \tag{23}$$

And we get:

$$\dot{\alpha}(t) = -\theta_0{}^2(1+\frac{1}{4}\theta_0{}^2)\frac{\sqrt{gl}}{4R_F}\sin(2t\sqrt{\frac{g}{l}}) - \frac{g}{R_F}(2-\frac{M}{m}\frac{R_G}{R_F}-\frac{\theta_0{}^4}{8})t \tag{24}$$

After integration by time, we get:

$$\alpha(t) = \alpha_0 + \frac{\theta_0{}^2 l}{8R_F}(1+\frac{1}{4}\theta_0{}^2)\cos(2t\sqrt{\frac{g}{l}}) - \frac{g}{2R_F}(2-\frac{\theta_0{}^4}{8}-\frac{M}{m}\frac{R_G}{R_F})t^2 \tag{25}$$

Finally, we will take:

$$\frac{g}{2R_F}t^2 = (t\sqrt{\frac{g}{2R_F}})^2 \approx \sin^2(t\sqrt{\frac{g}{2R_F}}) \tag{26}$$

For more simple calculation, we will take that it is:

$$2R_F = l;\ \omega = \sqrt{\frac{g}{l}} \tag{27}$$

Under the condition of:

$$\alpha(t)-\alpha_0 = \frac{\theta_0{}^4}{4}(1+\frac{1}{4}\theta_0{}^2)\cos(2\omega t) - (2-\frac{M}{m}\frac{R_G}{R_F}-\frac{\theta_0{}^4}{8})\sin^2(\omega t) \tag{28}$$

Since:

$$\cos(2\omega t) = \cos^2(\omega t) - \sin^2(\omega t) = 1-2\sin^2(\omega t) \tag{29}$$

It follows that:

$$\alpha(t)-\alpha_0 = \frac{\theta_0{}^2}{4}(1+\frac{1}{4}\theta_0{}^2) - (2-\frac{1}{2}\frac{M}{m}\frac{R_G}{l}+\frac{\theta_0{}^2}{2})\sin^2(t\sqrt{\frac{g}{l}}) \tag{30}$$

Function $\alpha(t)=\alpha_0$ has zeros which are:

$$\sin^2\left(t\sqrt{\frac{g}{l}}\right) = \frac{1}{4}\frac{\theta_0^{\,2}(1+\frac{1}{4}\theta_0^{\,2})}{2+\frac{\theta_0^{\,2}}{2}-\frac{1}{2}\frac{M}{m}\frac{R_G}{l}}$$

$$\sin\left(t\sqrt{\frac{g}{l}}\right) = \frac{\theta_0}{2}\sqrt{\frac{(1+\frac{1}{4}\theta_0^{\,2})}{2+\frac{\theta_0^{\,2}}{2}-\frac{1}{2}\frac{M}{m}\frac{R_G}{l}}}$$

Finally, zeros are:

$$t_K = \frac{\arcsin\left(\frac{\theta_0}{2}\sqrt{\frac{(1+\frac{\theta_0^{\,2}}{4})}{2+\frac{\theta_0^{\,2}}{2}-\frac{1}{2}\frac{M}{m}\frac{R_G}{l}}}\right)}{\sqrt{\frac{g}{l}}} + \frac{2\pi}{\sqrt{\frac{g}{l}}}k,\quad k=0,1,2...$$

In the moments t_K is $\alpha(t)-\alpha_0=0$ and the weight hits the surface.

Numbering:

$l = 0.4m$ $\qquad \omega = \sqrt{\frac{g}{l}} = 4.9523$

$R_G = 0.2m$ $\qquad M = m$ $\qquad \frac{M}{2m}\frac{R_G}{R_F} = \frac{0.1}{0.8} = 0.125$

$\frac{\theta_0^{\,2}}{2} = 0.2618$ $\qquad \frac{\theta_0^{\,2}}{4} = 0.0685$

$1+\frac{\theta_0^{\,2}}{4} = 1.0865$ $\qquad 2+\frac{\theta_0^{\,2}}{2}-\frac{M}{2m}\frac{R_G}{R_F} = 2.0121$

$\frac{\theta_0}{2} = 0.2618$ $\qquad X = \frac{\theta_0}{2}\sqrt{\frac{1+\frac{\theta_0^{\,2}}{4}}{2+\frac{\theta_0^{\,2}}{2}-\frac{M}{2m}\frac{R_G}{R_F}}} = 0.1908$

$\arcsin(X) = 0.192$

$$\boxed{t_K = 0.0425 + 0.6345k \quad k = 1,3,5...}$$

$t_0 = 0.0425$
$t_1 = 0.6770$
$t_2 = 1.3115$
$t_3 = 1.9460$
$t_4 = 2.5805$
$t_5 = 3.2150$
$t_6 = 3.8495$
$t_7 = 4.4850$
$t_8 = 5.1185$
$t_9 = 5.7530$

$t_{10} = 6.3875$
$t_{11} = 7.0220$
$t_{12} = 7.6565$
$t_{13} = 8.2910$
$t_{14} = 8.9255$
$t_{15} = 9.5600$
$t_{16} = 10.1945$
$t_{17} = 10.8290$
$t_{18} = 11.4635$
$t_{19} = 12.0980$

$t_{20} = 12.7325$
$t_{21} = 13.3670$
$t_{22} = 14.0015$
$t_{23} = 14.6360$
$t_{24} = 15.2705$
$t_{25} = 15.9050$
$t_{26} = 16.5395$
$t_{27} = 17.1740$
$t_{28} = 17.8085$
$t_{29} = 18.4430$

$t_{30} = 19.0775$
$t_{31} = 19.7120$

In Novi Sad, 2000

Academician professor Bratislav Tošić